

Hattori-Stallings trace and character

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Dedicated to the memory of Dieter Happel

Abstract

It is shown that Hattori-Stallings trace induces a homomorphism of abelian groups, called Hattori-Stallings character, from the K_1 -group of endomorphisms of the perfect derived category of an algebra to its zero-th Hochschild homology, which provides a new proof of Igusa-Liu-Paquette Theorem, i.e., the strong no loop conjecture for finite-dimensional elementary algebras, on the level of complexes. Moreover, the Hattori-Stallings traces of projective bimodules and one-sided projective bimodules are studied, which provides another proof of Igusa-Liu-Paquette Theorem on the level of modules.

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1 Introduction

Global dimension is a quite important homological invariant of an algebra or a ring. The (in)finiteness of global dimension plays an important role in representation theory of algebras. For instance, the bounded derived category of a finite-dimensional algebra has Auslander-Reiten triangles if and only if the algebra is of finite global dimension [10, 11]. There are some well-known conjectures related to the (in)finiteness of global dimension, such as no loop conjecture, Cartan determinant conjecture — the determinant of the Cartan

matrix of an artin ring of finite global dimension is 1 (ref. [5]), Hochschild homology dimension conjecture — a finite-dimensional algebra is of finite global dimension if and only if its Hochschild homology dimension is 0 (ref. [8]). To a finite-dimensional elementary algebra A , we can associate a quiver Q , called its Gabriel quiver (ref. [1, Page 65]). The (in)finiteness of the global dimension of A is closely related to the combinatorics of Q . If Q has no oriented cycles then $\text{gl.dim} A < \infty$ (ref. [4]). Obviously, its converse is not true in general. Nevertheless, if $\text{gl.dim} A < \infty$ then Q must have no loop, and 2-truncated cycle [3]. The former is due to the following conjecture:

No loop conjecture. *Let A be an artin algebra of finite global dimension. Then $\text{Ext}_A^1(S, S) = 0$ for every simple A -module S .*

The no loop conjecture was first explicitly established for artin algebras of global dimension two [6, Proposition]. For finite-dimensional elementary algebras, which is just the case that loop has its real geometric meaning, as shown in [13], this can be easily derived from an earlier result of Lenzing [16]. A stronger version of no loop conjecture is the following:

Strong no loop conjecture. *Let A be an artin algebra and S a simple A -module of finite projective dimension. Then $\text{Ext}_A^1(S, S) = 0$.*

The strong no loop conjecture is due to Zacharia [13], which is also listed as a conjecture in Auslander-Reiten-Smalø's book [1, Page 410, Conjecture (7)]. For finite-dimensional elementary algebras, and particularly, for finite-dimensional algebras over an algebraically closed field, it was proved in [14]. Some special cases were solved in [7, 15, 17, 18, 19, 21].

In this paper, we shall show that Hattori-Stallings trace induces a homomorphism of abelian groups, called Hattori-Stallings character, from the K_1 -group of endomorphisms of the perfect derived category of an algebra to its zero-th Hochschild homology (see Section 2), which provides a neat proof of Igusa-Liu-Paquette Theorem, i.e., the strong no loop conjecture for finite-dimensional elementary algebras, on the level of complexes (see Section 3). Moreover, in Section 4, we shall study the Hattori-Stallings traces of projective bimodules and one-sided projective bimodules, which provides a simpler proof of Igusa-Liu-Paquette Theorem on the level of modules. A key point is the bimodule characterization of the projective dimension of a simple module.

2 Hattori-Stallings character

In this section, we shall show that Hattori-Stallings trace induces a homomorphism of abelian groups, called Hattori-Stallings character, from the K_1 -group of endomorphisms of the perfect derived category of an algebra to its zero-th Hochschild homology.

2.1 Hattori-Stallings traces

Let A be a ring with identity. Denote by $\text{Mod}A$ the category of right A -modules, and by $\text{proj}A$ the full subcategory of $\text{Mod}A$ consisting of all finitely generated projective right A -modules. Denote by $D(A)$ the unbounded derived categories of the complexes of right A -modules, and by $K^b(\text{proj}A)$ the homotopy category of the bounded complexes of finitely generated projective right A -modules, which is triangle equivalent to the perfect derived category of A .

For each $P \in \text{proj}A$, there is an isomorphism of abelian groups

$$\phi_P : P \otimes_A \text{Hom}_A(P, A) \rightarrow \text{End}_A(P)$$

defined by $\phi_P(p \otimes f)(p') = pf(p')$ for all $p, p' \in P$ and $f \in \text{Hom}_A(P, A)$. There is also a homomorphism of abelian groups

$$\psi_P : P \otimes_A \text{Hom}_A(P, A) \rightarrow A/[A, A]$$

defined by $\psi_P(p \otimes f) = \overline{f(p)}$ for all $p \in P$ and $f \in \text{Hom}_A(P, A)$. Here, $[A, A]$ is the additive subgroup of A generated by all commutators $[a, b] := ab - ba$ with $a, b \in A$. It is well-known that the abelian group $A/[A, A]$ is isomorphic to the zero-th Hochschild homology group $HH_0(A)$ of A . The homomorphism of abelian groups

$$\text{tr}_P := \psi_P \phi_P^{-1} : \text{End}_A(P) \rightarrow A/[A, A]$$

is called *the Hattori-Stallings trace of P* .

Hattori-Stallings trace has the following properties:

Proposition 1. (Hattori [12], Stallings [20], Lenzing [16]) *Let $P, P', P'' \in \text{proj}A$.*

(HS1) *If $f \in \text{End}_A(P)$ and $g \in \text{Hom}_A(P, P')$ is an isomorphism then $\text{tr}_P(f) = \text{tr}_{P'}(gfg^{-1})$.*

(HS2) *If $f, f' \in \text{End}_A(P)$ then $\text{tr}_P(f + f') = \text{tr}_P(f) + \text{tr}_P(f')$.*

- (HS3) If $f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \in \text{End}_A(P \oplus P')$ then $\text{tr}_{P \oplus P'}(f) = \text{tr}_P(f_{11}) + \text{tr}_{P'}(f_{22})$.
- (HS4) If $f \in \text{Hom}_A(P, P')$ and $g \in \text{Hom}_A(P', P)$ then $\text{tr}_P(gf) = \text{tr}_{P'}(fg)$.
- (HS5) If

$$\begin{array}{ccccccc} 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' \longrightarrow 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' \\ 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' \longrightarrow 0 \end{array}$$

is a commutative diagram with exact rows then $\text{tr}_P(f) = \text{tr}_{P'}(f') + \text{tr}_{P''}(f'')$.

- (HS6) If $l_a \in \text{End}_A(A)$ is the left multiplication by $a \in A$ then $\text{tr}_A(l_a) = \bar{a}$, the equivalence class of a in $A/[A, A]$.

2.2 K_1 -groups of endomorphisms

Let \mathcal{C} be a category. Denote by $\text{end}\mathcal{C}$ the category of endomorphisms of \mathcal{C} , whose objects are all pairs (C, f) with $C \in \mathcal{C}$ and $f \in \text{End}_{\mathcal{C}}(C)$ and whose Hom sets are $\text{Hom}_{\text{end}\mathcal{C}}((C, f), (C', f')) := \{g \in \text{Hom}_{\mathcal{C}}(C, C') \mid gf = f'g\}$. Obviously, if \mathcal{C} is a skeletally small category then so is $\text{end}\mathcal{C}$.

For a skeletally small triangulated category \mathcal{T} , we define its K_1 -group of endomorphisms (cf. [2, Chapter III]), denoted by $K_1(\text{end}\mathcal{T})$, to be the factor group of the free abelian group generated by all isomorphism classes of objects in $\text{end}\mathcal{T}$ modulo the relations:

- (K1) $[(T, f + f')] = [(T, f)] + [(T, f')]$ for all $T \in \mathcal{T}$ and $f, f' \in \text{End}_{\mathcal{T}}(T)$.
- (K2) $[(T, f)] = [(T', f')] + [(T'', f'')]$ for every commutative diagram

$$\begin{array}{ccccccc} T' & \longrightarrow & T & \longrightarrow & T'' & \longrightarrow & \\ \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ T' & \longrightarrow & T & \longrightarrow & T'' & \longrightarrow & \end{array}$$

with triangles as rows.

Clearly, if two skeletally small triangulated categories are triangle equivalent then their K_1 -groups of endomorphisms are isomorphic.

2.3 Hattori-Stallings character

For any ring A with identity, both the exact category $\text{proj}A$ and the triangulated category $K^b(\text{proj}A)$ are skeletally small. So is $\text{end}K^b(\text{proj}A)$.

The main result in this section is the following:

Theorem 1. *Let A be a ring with identity. Then the map*

$$\mathrm{tr} : K_1(\mathrm{end}K^b(\mathrm{proj}A)) \rightarrow A/[A, A], \quad [(P^\bullet, \overline{f^\bullet})] \mapsto \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}_{P^i}(f^i),$$

is a homomorphism of abelian groups, called the Hattori-Stallings character of A , which satisfies the trace property (TP):

$$\mathrm{tr}([(P^\bullet, \overline{g^\bullet} \circ \overline{f^\bullet})]) = \mathrm{tr}([(P'^\bullet, \overline{f'^\bullet} \circ \overline{g'^\bullet})])$$

for all $\overline{f^\bullet} \in \mathrm{Hom}_{K^b(\mathrm{proj}A)}(P^\bullet, P'^\bullet)$ and $\overline{g^\bullet} \in \mathrm{Hom}_{K^b(\mathrm{proj}A)}(P'^\bullet, P^\bullet)$.

Proof. Since $P^\bullet \in K^b(\mathrm{proj}A)$, P^i is zero for almost all $i \in \mathbb{Z}$. Thus the sum $\sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}_{P^i}(f^i)$ makes sense.

Step 1. $\sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}_{P^i}(f^i)$ is independent of the choice of the representative f^\bullet of the homotopy equivalence class $\overline{f^\bullet}$. Indeed, if $\overline{f^\bullet} = \overline{f'^\bullet}$, then $f^\bullet - f'^\bullet = s^{\bullet+1}d^\bullet + d^{\bullet-1}s^\bullet$ for some homotopy map s^\bullet , where d^\bullet is the differential of P^\bullet . It follows from (HS2) and (HS4) that

$$\begin{aligned} & \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}_{P^i}(f^i) - \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}_{P^i}(f'^i) \\ &= \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}_{P^i}(f^i - f'^i) \\ &= \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}_{P^i}(s^{i+1}d^i + d^{i-1}s^i) \\ &= \sum_{i \in \mathbb{Z}} (-1)^i (\mathrm{tr}_{P^i}(s^{i+1}d^i) + \mathrm{tr}_{P^i}(d^{i-1}s^i)) \\ &= \sum_{i \in \mathbb{Z}} (-1)^i (\mathrm{tr}_{P^i}(s^{i+1}d^i) + \mathrm{tr}_{P^{i-1}}(s^i d^{i-1})) = 0. \end{aligned}$$

Step 2. $\sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}_{P^i}(f^i)$ is also independent of the choice of the representative $(P^\bullet, \overline{f^\bullet})$ of the isomorphism class $[(P^\bullet, \overline{f^\bullet})]$. Indeed, if $(P^\bullet, \overline{f^\bullet}) \cong (P'^\bullet, \overline{f'^\bullet})$ then there are morphisms $\overline{g^\bullet} \in \mathrm{Hom}_{K^b(\mathrm{proj}A)}(P^\bullet, P'^\bullet)$ and $\overline{g'^\bullet} \in \mathrm{Hom}_{K^b(\mathrm{proj}A)}(P'^\bullet, P^\bullet)$ such that $\overline{g'^\bullet} \circ \overline{g^\bullet} = \overline{1}$, $\overline{g^\bullet} \circ \overline{g'^\bullet} = \overline{1}$, and $\overline{f'^\bullet} \circ \overline{g^\bullet} = \overline{g^\bullet} \circ \overline{f^\bullet}$. Thus $\overline{f^\bullet} = \overline{g'^\bullet} \circ \overline{g^\bullet} \circ \overline{f^\bullet}$ and $\overline{g^\bullet} \circ \overline{f^\bullet} \circ \overline{g'^\bullet} = \overline{f'^\bullet} \circ \overline{g^\bullet} \circ \overline{g'^\bullet} = \overline{f'^\bullet}$. It follows from Step 1 and (HS4) that

$$\begin{aligned} \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}_{P^i}(f^i) &= \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}_{P^i}(g'^i g^i f^i) \\ &= \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}_{P'^i}(g^i f^i g'^i) \\ &= \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}_{P'^i}(f'^i g^i g'^i) \\ &= \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}_{P'^i}(f'^i). \end{aligned}$$

Now we have shown that tr is well-defined on the free abelian group generated by the isomorphism classes of $\mathrm{end}K^b(\mathrm{proj}A)$.

Step 3. $\mathrm{tr}([(P^\bullet, \overline{f^\bullet} + \overline{f'^\bullet})]) = \mathrm{tr}([(P^\bullet, \overline{f^\bullet})]) + \mathrm{tr}([(P^\bullet, \overline{f'^\bullet})])$ for all $P^\bullet \in K^b(\mathrm{proj}A)$ and $\overline{f^\bullet}, \overline{f'^\bullet} \in \mathrm{End}_{K^b(\mathrm{proj}A)}(P^\bullet)$. Indeed, this is clear by (HS2).

Step 4. $\text{tr}([(P^\bullet, \overline{f^\bullet})]) = \text{tr}([(P'^\bullet, \overline{f'^\bullet})]) + \text{tr}([(P''^\bullet, \overline{f''^\bullet})])$ for every commutative diagram

$$\begin{array}{ccccc} P'^\bullet & \xrightarrow{\overline{u^\bullet}} & P^\bullet & \xrightarrow{\overline{v^\bullet}} & P''^\bullet \longrightarrow \\ \downarrow \overline{f'^\bullet} & & \downarrow \overline{f^\bullet} & & \downarrow \overline{f''^\bullet} \\ P'^\bullet & \xrightarrow{\overline{u^\bullet}} & P^\bullet & \xrightarrow{\overline{v^\bullet}} & P''^\bullet \longrightarrow \end{array}$$

with triangles as rows. Indeed, in $K^b(\text{proj} A)$ each triangle $P'^\bullet \xrightarrow{\overline{u^\bullet}} P^\bullet \xrightarrow{\overline{v^\bullet}} P''^\bullet \longrightarrow$ is isomorphic to a triangle

$$P'^\bullet \xrightarrow{\begin{bmatrix} \overline{1} \\ 0 \end{bmatrix}} \text{Cyl}(u^\bullet) \xrightarrow{\begin{bmatrix} 0 & \overline{1} \end{bmatrix}} \text{Cone}(u^\bullet) \longrightarrow$$

where $\text{Cyl}(u^\bullet)$ and $\text{Cone}(u^\bullet)$ are the cylinder and cone of the cochain map $u^\bullet : P'^\bullet \longrightarrow P^\bullet$ respectively. Thus, by Step 2, it is enough to consider the case that the following diagram

$$\begin{array}{ccccccc} P'^\bullet & \xrightarrow{\begin{bmatrix} \overline{1} \\ 0 \end{bmatrix}} & P'^\bullet \oplus P''^\bullet & \xrightarrow{\begin{bmatrix} 0 & \overline{1} \end{bmatrix}} & P''^\bullet & \longrightarrow & \\ \downarrow \overline{f'^\bullet} & & \downarrow \begin{bmatrix} \overline{f'^\bullet} & \overline{f''^\bullet} \\ 0 & \overline{f''^\bullet} \end{bmatrix} & & \downarrow \overline{f''^\bullet} & & \\ P'^\bullet & \xrightarrow{\begin{bmatrix} \overline{1} \\ 0 \end{bmatrix}} & P'^\bullet \oplus P''^\bullet & \xrightarrow{\begin{bmatrix} 0 & \overline{1} \end{bmatrix}} & P''^\bullet & \longrightarrow & \end{array}$$

with triangles as rows is commutative. In this case, by (HS3), we have $\sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P'^i}(f'^i) + \sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P''^i}(f''^i) = \sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P'^i \oplus P''^i} \begin{pmatrix} f'^i & f''^i \\ 0 & f''^i \end{pmatrix}$.

Now we have shown that the Hattori-Stallings character tr is well-defined. Next, we prove that it satisfies trace property (TP).

Step 5. $\text{tr}([(P^\bullet, \overline{g^\bullet} \circ \overline{f^\bullet})]) = \text{tr}([(P'^\bullet, \overline{f'^\bullet} \circ \overline{g'^\bullet})])$ for all morphisms $\overline{f^\bullet} \in \text{Hom}_{K^b(\text{proj} A)}(P^\bullet, P'^\bullet)$ and $\overline{g^\bullet} \in \text{Hom}_{K^b(\text{proj} A)}(P'^\bullet, P^\bullet)$. Indeed, this is clear by Step 2 and (HS4). \square

3 Igusa-Liu-Paquette Theorem

In this section, we shall apply Hattori-Stallings character to give a new proof of Igusa-Liu-Paquette Theorem on the level of complexes. From now

on, let k be a field and A a finite-dimensional elementary k -algebra, i.e., $A/J \cong k^n$ for some natural number n , where J denotes the Jacobson radical of A .

3.1 Projective dimension

Some homological properties on modules can be characterized by those of bimodules. For instance, Happel showed that for a finite-dimensional k -algebra A , $\text{gl.dim} A = \text{pd}_{A^e} A$ (ref. [9]). In this subsection, we shall give a bimodule characterization of the projective dimension of a simple module. For this, we need the following well-known result, which implies that $\text{top} A = A/J$ is a “testing module” of the projective dimension of an A -module:

Lemma 1. *Let A be an artin algebra, and $M \neq 0$ a finitely generated left A -module. Then $\text{pd}_A M = \sup\{i | \text{Ext}_A^i(M, A/J) \neq 0\} = \sup\{i | \text{Tor}_i^A(A/J, M) \neq 0\}$.*

Proof. Let P^\bullet be a minimal projective resolution of the left A -module M . Then all the differentials of the complex $\text{Hom}_A(P^\bullet, A/J)$ are zero. Thus $\text{Ext}_A^i(M, A/J) = \text{Hom}_A(P^{-i}, A/J)$. Hence, $\text{pd}_A M = \sup\{i | P^{-i} \neq 0\} = \sup\{i | \text{Hom}_A(P^{-i}, A/J) \neq 0\} = \sup\{i | \text{Ext}_A^i(M, A/J) \neq 0\}$.

Similarly, all the differentials of the complex $A/J \otimes_A P^\bullet$ are zero. Thus $\text{Tor}_i^A(A/J, M) = A/J \otimes_A P^{-i}$. Therefore, $\text{pd}_A M = \sup\{i | P^{-i} \neq 0\} = \sup\{i | A/J \otimes_A P^{-i} \neq 0\} = \sup\{i | \text{Tor}_i^A(A/J, M) \neq 0\}$. \square

A key point of this paper is the following observation:

Lemma 2. *Let A be a finite-dimensional elementary k -algebra, $S = Ae/Je$ the left simple A -module corresponding to a primitive idempotent e in A , and $\bar{A} := A/A(1 - e)A$. Then $\text{pd}_A S = \text{pd}_{A \otimes_k \bar{A}^{\text{op}}} \bar{A}$.*

Proof. We have isomorphisms $\text{Tor}_i^A(A/J, S) \cong H^{-i}(A/J \otimes_A^L S) \cong H^{-i}(A/J \otimes_A^L \bar{A} \otimes_{\bar{A}}^L S) \cong H^{-i}((A/J \otimes_k S) \otimes_{A \otimes_k \bar{A}^{\text{op}}}^L \bar{A}) \cong \text{Tor}_i^{A \otimes_k \bar{A}^{\text{op}}}(A/J \otimes_k S, \bar{A})$. Applying Lemma 1 twice, we obtain $\text{pd}_A S = \text{pd}_{A \otimes_k \bar{A}^{\text{op}}} \bar{A}$, since $A/J \otimes_k S = \text{top}(A \otimes_k \bar{A}^{\text{op}})$. \square

3.2 A new proof of Igusa-Liu-Paquette Theorem

Theorem 2. (Igusa-Liu-Paquette [14]) *Let A be a finite-dimensional elementary k -algebra, S a left simple A -module, and $\text{pd}_A S < \infty$. Then $\text{Ext}_A^1(S, S) = 0$.*

Proof. We may assume that S is the left simple A -module Ae/Je corresponding to a primitive idempotent e in A and $\bar{A} := A/A(1-e)A$.

We have the following commutative diagram in $D(A)$:

$$\begin{array}{ccccccc} J^{j+1} & \longrightarrow & J^j & \longrightarrow & J^j/J^{j+1} & \longrightarrow & \\ \downarrow l_a & & \downarrow l_a & & \downarrow l_a=0 & & \\ J^{j+1} & \longrightarrow & J^j & \longrightarrow & J^j/J^{j+1} & \longrightarrow & \end{array}$$

with triangles as rows for all $a \in J$, the Jacobson radical of A , and $0 \leq j \leq t-1$ where t is the Loewy length of A . Applying the derived tensor functor $-\otimes_A^L \bar{A}$ to the commutative diagram above, we obtain the following commutative diagram in $D(\bar{A})$:

$$\begin{array}{ccccccc} J^{j+1} \otimes_A^L \bar{A} & \longrightarrow & J^j \otimes_A^L \bar{A} & \longrightarrow & (J^j/J^{j+1}) \otimes_A^L \bar{A} & \longrightarrow & \\ \downarrow l_a \otimes_A^L \bar{A} & & \downarrow l_a \otimes_A^L \bar{A} & & \downarrow l_a \otimes_A^L \bar{A}=0 & & \\ J^{j+1} \otimes_A^L \bar{A} & \longrightarrow & J^j \otimes_A^L \bar{A} & \longrightarrow & (J^j/J^{j+1}) \otimes_A^L \bar{A} & \longrightarrow & \end{array}$$

with triangles as rows for all $a \in J$ and $0 \leq j \leq t-1$. By the assumption $\text{pd}_A S < \infty$ and Lemma 2, we have a bounded finitely generated projective A - \bar{A} -bimodules resolution P^\bullet of \bar{A} . Thus we have the following commutative diagram in $K^b(\text{proj } \bar{A})$:

$$\begin{array}{ccccccc} J^{j+1} \otimes_A P^\bullet & \longrightarrow & J^j \otimes_A P^\bullet & \longrightarrow & (J^j/J^{j+1}) \otimes_A P^\bullet & \longrightarrow & \\ \downarrow l_a & & \downarrow l_a & & \downarrow l_a=0 & & \\ J^{j+1} \otimes_A P^\bullet & \longrightarrow & J^j \otimes_A P^\bullet & \longrightarrow & (J^j/J^{j+1}) \otimes_A P^\bullet & \longrightarrow & \end{array}$$

with triangles as rows for all $a \in J$ and $0 \leq j \leq t-1$. Therefore, for any $\bar{a} \in \bar{J}$, the Jacobson radical of \bar{A} , the equivalence class of \bar{a} in $\bar{A}/[\bar{A}, \bar{A}]$

$$\begin{aligned} \bar{a} = \text{tr}([\bar{A}, l_{\bar{a}}]) &= \text{tr}([\bar{A}, l_a]) &= \text{tr}([(J^0 \otimes_A P^\bullet, l_a)]) \\ &= \text{tr}([(J^1 \otimes_A P^\bullet, l_a)]) \\ &= \dots \\ &= \text{tr}([(J^t \otimes_A P^\bullet, l_a)]) \\ &= \text{tr}([(0, 0)]) = 0. \end{aligned}$$

Hence, $\bar{J} \subseteq [\bar{A}, \bar{A}]$.

Let $A' := \bar{A}/\bar{J}^2$ and $J' = \bar{J}/\bar{J}^2$ its Jacobson radical. Then A' is a local algebra with radical square zero, and thus commutative. Since $\bar{J} \subseteq [\bar{A}, \bar{A}]$, we have $J' \subseteq [A', A'] = 0$, i.e., $J' = 0$. Hence, $\text{Ext}_A^1(S, S) \cong eJe/eJ^2e \cong J' = 0$. \square

4 Hattori-Stallings traces of bimodules

In this section, we shall study the Hattori-Stallings traces of projective bimodules and one-sided projective bimodules, which provides another proof of Igusa-Liu-Paquette Theorem on the level of modules.

Firstly, we consider the Hattori-Stallings traces of finitely generated projective bimodules.

Proposition 2. *Let A and B be finite-dimensional k -algebras, and P a finitely generated projective A - B -bimodule. Then $\text{tr}_{P_B}(l_a) = 0$ for all $a \in J$, the Jacobson radical of A .*

Proof. We have the following commutative diagram in $\text{Mod}A$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & J^{j+1} & \longrightarrow & J^j & \longrightarrow & J^j/J^{j+1} \longrightarrow 0 \\ & & \downarrow l_a & & \downarrow l_a & & \downarrow l_a=0 \\ 0 & \longrightarrow & J^{j+1} & \longrightarrow & J^j & \longrightarrow & J^j/J^{j+1} \longrightarrow 0 \end{array}$$

with exact rows for all $a \in J$ and $0 \leq j \leq t-1$ where t is the Loewy length of A . Since P is a finitely generated projective A - B -bimodule, we have the following commutative diagram in $\text{proj}B$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & J^{j+1} \otimes_A P & \longrightarrow & J^j \otimes_A P & \longrightarrow & (J^j/J^{j+1}) \otimes_A P \longrightarrow 0 \\ & & \downarrow l_a & & \downarrow l_a & & \downarrow l_a=0 \\ 0 & \longrightarrow & J^{j+1} \otimes_A P & \longrightarrow & J^j \otimes_A P & \longrightarrow & (J^j/J^{j+1}) \otimes_A P \longrightarrow 0 \end{array}$$

with exact rows for all $a \in J$ and $0 \leq j \leq t-1$. It follows from (HS5) that $\text{tr}_{P_B}(l_a) = \text{tr}_{J^0 \otimes_A P_B}(l_a) = \text{tr}_{J^1 \otimes_A P_B}(l_a) = \cdots = \text{tr}_{J^{t-1} \otimes_A P_B}(l_a = 0) = 0$ for all $a \in J$. \square

Secondly, we consider the Hattori-Stallings traces of finitely generated one-sided projective bimodules.

Proposition 3. *Let A and B be finite-dimensional k -algebras, M a finitely generated A - B -bimodule which is projective as a right B -module, and P^\bullet a finitely generated projective A - B -bimodule resolution of M . Then*

$$\text{tr}_{M_B}(l_a) = (-1)^i \text{tr}_{\Omega_i(M)}(l_a)$$

for all $a \in J$ and $i \in \mathbb{N}$, where $\Omega_i(M)$ is the i -th syzygy of M on P^\bullet .

Proof. Since M_B is projective, all $\Omega_i(M)_B$'s are projective. We have the following commutative diagrams in $\text{proj} B$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_i(M) & \longrightarrow & P^{-i+1} & \longrightarrow & \Omega_{i-1}(M) \longrightarrow 0 \\ & & \downarrow l_a & & \downarrow l_a & & \downarrow l_a \\ 0 & \longrightarrow & \Omega_i(M) & \longrightarrow & P^{-i+1} & \longrightarrow & \Omega_{i-1}(M) \longrightarrow 0 \end{array}$$

with exact rows for all $a \in J$ and $i \geq 1$. By Proposition 2 and (HS5), we obtain $\text{tr}_{\Omega_i(M)}(l_a) = -\text{tr}_{\Omega_{i-1}(M)}(l_a)$, thus $\text{tr}_{M_B}(l_a) = \text{tr}_{\Omega_0(M)}(l_a) = -\text{tr}_{\Omega_1(M)}(l_a) = \cdots = (-1)^i \text{tr}_{\Omega_i(M)}(l_a)$ for all $a \in J$ and $i \in \mathbb{N}$. \square

Finally, we provide another proof of Igusa-Liu-Paquette Theorem, i.e., Theorem 2, on the level of modules.

Proof. By the assumption $\text{pd}_A S < \infty$ and Lemma 2, we have $\text{pd}_{A \otimes_k \bar{A}^{\text{op}}} \bar{A} < \infty$. It follows from Proposition 3 that, for any $\bar{a} \in \bar{J}$, the equivalence class of \bar{a} in $\bar{A}/[\bar{A}, \bar{A}]$, $\bar{a} = \text{tr}_{\bar{A}}(l_{\bar{a}}) = \text{tr}_{\bar{A}}(l_a) = (-1)^i \text{tr}_{\Omega_i(M)}(l_a)$ which equals 0 for $i > \text{pd}_{A \otimes_k \bar{A}^{\text{op}}} \bar{A}$. Thus $\bar{J} \subseteq [\bar{A}, \bar{A}]$. Then we may continue as the last paragraph of the proof of Theorem 2 in Section 3.2. \square

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References

- [1] M. Auslander, I. Reiten and S.O. Smalø, Representation Theory of Artin Algebras, Cambridge Studies in Advanced Mathematics 36, Cambridge University Press, Cambridge, 1995.
- [2] H. Bass, K-theory and stable algebra, I.H.E.S. Publ. Math. 22 (1964), 5–60.
- [3] P.A. Bergh, Y. Han and D. Madsen, Hochschild homology and truncated cycles, Proc. Amer. Math. Soc. 140 (2012), 1133–1139.
- [4] S. Eilenberg, H. Nagao and T. Nakayama, On the dimension of modules and algebras. IV. Dimension of residue rings of hereditary rings, Nagoya Math. J. 10 (1956), 87–95.
- [5] K.R. Fuller, The Cartan determinant and global dimension of Artinian rings, Contemp. Math. 124 (1992), 51–72.
- [6] E.L. Green, W.H. Gustafson and D. Zacharia, Artin rings of global dimension two, J. Algebra 92 (1985), 375–379.
- [7] E.L. Green, Ø. Solberg and D. Zacharia, Minimal projective resolutions, Trans. Amer. Math. Soc. 353 (2001), 2915–2939.
- [8] Y. Han, Hochschild (co)homology dimension. J. London Math. Soc. 73 (2006), 657–668.

- [9] D. Happel, Hochschild cohomology of finite-dimensional algebras, in: Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988), Lecture Notes in Math., Vol. 1404, Springer, Berlin, 1989, pp. 108–126.
- [10] D. Happel, On the derived category of a finite dimensional algebra, *Comment. Math. Helv.* 62 (1987) 339–389.
- [11] D. Happel, Auslander-Reiten triangles in derived categories of finite-dimensional algebras, *Proc. Amer. Math. Soc.* 112 (1991) 641–648.
- [12] A. Hattori, Rank element of a projective module, *Nagoya Math. J.* 25 (1965), 113–120.
- [13] K. Igusa, Notes on the no loops conjecture, *J. Pure Appl. Algebra* 69 (1990), 161–176.
- [14] K. Igusa, S.P. Liu and C. Paquette, A proof of the strong no loop conjecture, *Adv. Math.* 228 (2011), 2731–2742.
- [15] B.T. Jensen, Strong no-loop conjecture for algebras with two simples and radical cube zero, *Colloq. Math.* 102 (2005), 1–7.
- [16] H. Lenzing, Nilpotente elemente in ringen von endlicher globaler dimension, *Math. Z.* 108 (1969), 313–324.
- [17] S. Liu and J.P. Morin, The strong no loop conjecture for special biserial algebras, *Proc. Amer. Math. Soc.* 132 (2004), 3513–3523.
- [18] N. Marmaridis and A. Papistas, Extensions of abelian categories and the strong no-loops conjecture, *J. Algebra* 178 (1995), 1–20.
- [19] D. Skorodumov, The strong no loop conjecture for mild algebras, *J. Algebra* 336 (2011), 301–320.
- [20] J. Stallings, Centerless groups — an algebraic formulation of Gottlieb's theorem, *Topology* 4 (1965), 129–134.
- [21] D. Zacharia, Special monomial algebras of finite global dimension, in: Perspectives in Ring Theory, Proc. NATO Adv. Res. Workshop, Antwerp/Belg. 1987, in: NATO ASI Ser. C, vol. 233, 1988, pp. 375–378.